Spinning Particle Dynamics on Six-Dimensional Minkowski Space

S. L. Lyakhovich, A. A. Sharapov and K. M. Shekhter

Department of Physics, Tomsk State University,
Lenin Ave. 36, Tomsk 634050, Russia

Abstract

Massive spinning particle in $6d$-Minkowski space is described as a mechanical system with the configuration space $R^{5,1} \times CP^3$. The action functional of the model is unambiguously determined by the requirement of identical (off-shell) conservation for the phase-space counterparts of three Casimir operators of Poincaré group. The model proves to be completely solvable. Its generalization to the constant curvature background is presented. Canonical quantization of the theory leads to the relativistic wave equations for the irreducible $6d$ fields.

1 Introduction

A classical description of relativistic spinning particles is one of the traditional branches of theoretical physics having a long history [1, 2, 3]. By now, several approaches to this problem have been developed. Most of the researches are based on the enlargement of the Minkowski space by extra variables, anticommuting [2] or commuting [3, 4, 5], responsible for the spin evolution. Being well adapted for the quantization, the theories using Grassmann variables encounter, however, difficulties on attempting to justify them at the classical level. Besides that, the quantization of these theories lead to the Poincaré representation of fixed spin.

The orbit method, developed in [6], is the universal approach for the description of the elementary systems. The basic object of this approach is a presymplectic manifold $\mathcal{E}$, being a homogeneous transformation space for a certain Lie group G, and the system is considered as "elementary" for this group. The manifold carries the invariant and degenerate closed two-form Ω such that quotient space $\mathcal{E}/\ker\Omega$ is a homogeneous symplectic manifold (in fact, it may be identified with some covering space for coadjoint orbit $\mathcal{O}$ of group G). If θ is a potential one–form for Ω then the first-order action functional of the system may be written as

$$S = \int \theta$$

Being applied to the Poincaré group, this method gives the Souriau classification of the spinning particles. Meanwhile, there is another trend to describe a spinning particle by means of a traditional formalism based on an appropriate choice of the configuration space for spin [1-5].

In a recent paper [4], the new model was proposed for a massive particle of arbitrary spin in $d = 4$ Minkowski space to be a mechanical system with the configuration space $R^{3,1} \times S^2$, where two sphere S^2 corresponds to the spinning degrees of freedom. It was shown that principles underlying the model have simple physical and geometrical origin. Quantization of the model leads to the unitary massive representations of the Poincaré group. The model allows the direct extension to the case of higher superspin superparticle and the generalization to the anti-de Sitter space.

Despite the apparent simplicity of the model's construction, its higher dimensional generalization is not so evident, and the most crucial point is the choice of configuration space for spin. In this work we describe the massive spinning particle in six-dimensional Minkowski space $R^{5,1}$, that may be considered as a first step towards the uniform model construction for all higher dimensions. It should also be noted that this generalization may have a certain interest in its own rights since six is the one in every four dimensions: 3, 4, 6 and 10 possess the remarkable properties such as the presence of two-component spinor formalism or light-likeness of the spinor bilinear [9]. These properties are conditioned by the connection between the division algebras and the Lorentz groups of these spaces [8]

$$SL\,(2, A) \sim SO^{\uparrow}\,(\dim A + 1, 1)$$

where A are the division algebras R, C, H, O of real and complex numbers, quaternions and octonions respectively. Besides that, these are exactly the dimensions where the classical theory of Green-Schwarz superstring can be formulated [7].

Let us now sketch the broad outlines of the construction. First of all, for any even dimension d, the model's configuration space is chosen to be the direct product of Minkowski space $R^{d-1,1}$ and some m-dimensional compact manifold K^m being a homogeneous transformation space for the Lorentz group $SO(d-1,1)$. Then the manifold $M^{d+m} = R^{d-1,1} \times K^m$ proves to be the homogeneous transformation space for the Poincaré group. The action of the Poincaré group on M^{d+m} is unambiguously lifted up to the action on the cotangent bundle $T^*(M^{d+m})$ being the extended phase space of the model. It is well known that the massive unitary irreducible representations of the Poincaré group are uniquely characterized by the eigenvalues of $d/2$ Casimir operators

$$C_1 = \mathbf{P}^2 \ , \ C_{i+1} = \mathbf{W}^{A_1 \ldots A_{2i-1}} \mathbf{W}_{A_1 \ldots A_{2i-1}} \ , \quad i = 1, \ldots, \frac{d-2}{2} \ ,$$

where $\mathbf{W}_{A_1 \ldots . A_{2i-1}} = \epsilon_{A_1 \ldots A_d} \mathbf{J}^{A_{2i} A_{2i+1}} \ldots \mathbf{J}^{A_{d-2} A_{d-1}} \mathbf{P}^{A_d}$ and $\mathbf{J}_{AB}, \mathbf{P}_C$ are the Poincaré generators. This leads us to require the identical (off-shell) conservation for the quantum numbers associated with the phase space counterparts of Casimir operators. In other words $d/2$ first-class constraints should appear in the theory.

Finally, the dimensionality m of the manifold K^m is specified from the condition that the reduced (physical) phase space of the model should be a homogeneous symplectic manifold of Poincaré group (in fact, it should coincide with the coadjoint orbit of maximal dimension $d^2/2$). A simple calculation leads to $m = d(d-2)/4$. In the case of four-dimensional Minkowski space this yields $m = 2$ and two-sphere S^2 turns out to be the unique candidate for the internal space of the spinning degrees of freedom. In the case considered in this paper $d = 6$, and hence $m = 6$. As will be shown below the suggestive choice for K^6 is the complex projective space CP^3.

The models can be covariantly quantized a lâ Dirac by imposing the first-class constraints on the physical states being the smooth complex functions on the homogeneous space $M^{d(d+2)/4} = R^{d-1,1} \times K^{d(d-2)/4}$

$$(\widehat{C}_i - \delta_i)\Psi = 0 \quad , \qquad i = 1, \ldots, \frac{d}{2} \ ,$$

where the parameters δ_i are the quantum numbers characterizing the massive unitary representation of the Poincaré group. Thus the quantization of the spinning particle theories reduces to the standard mathematical problem of harmonic analysis on homogeneous spaces. It should be remarked that manifold $M^{d(d+2)/4}$ may be thought of as the *minimal* (in sense of its dimensionality) one admitting a non-trivial dynamics of arbitrary spin, and hence it is natural to expect that the corresponding Hilbert space of physical states will carry the *irreducible* representation of the Poincaré group.

The paper is organized as follows. Sec.2 deals with the description of the configuration space geometry, its local structure and various parametrizations. In sec.3 we derive the model's action functional in the first order formalism. We also consider the solutions of classical equations of motion and discuss the geometry of the trajectories. In sec.4 the second order formalism for the theory is presented and the different reduced forms of Lagrangian are discussed. Here we also investigate the causality conditions for the theory. Sec.5 is devoted to the quantization of the theory in the Hilbert space of smooth tensor fields over M^{12}. The connection with the relativistic wave equations is apparently stated. In the conclusion we discuss the received results and further perspectives. In the Appendix we have collected the basic facts of half spinorial formalism in six-dimensions.

2 Geometry of the configuration space and covariant parametrization

We start with describing a covariant realization for the model's configuration space chosen as $M^{12} = R^{5,1} \times CP^3$. The manifold M^{12} is the homogeneous transformation space for Poincaré group P and, hence, it can be realized as a coset space P/H for some subgroup $H \subset P$. In order to present the subgroup H in an explicit form it is convenient to make Iwasawa decomposition for six-dimensional Lorentz group $SO\,(5,1)$ in maximal compact subgroup $SO\,(5)$ and solvable factor R

$$SO\,(5,1) = SO\,(5)\,R \tag{1}$$

Then the minimal parabolic subgroup, being defined as normalizer of R in $SO\,(5,1)$, coincides with $SO\,(4)\,R$. By means of the decomposition $SO\,(4) = SO\,(3) \times SO\,(3)$ the subgroup H is identified with $[SO\,(2) \times SO\,(3)]\,R$. Thus

$$M^{12} = R^{5,1} \times CP^3 \sim \frac{Poincar\acute{e}\,group}{[SO\,(2) \times SO\,(3)]\,R} \sim R^{5,1} \times \frac{SO\,(5)}{SO\,(2) \times SO\,(3)} \tag{2}$$

and thereby one has the isomorphism

$$CP^3 \sim \frac{SO\,(5)}{SO\,(2) \times SO\,(3)} \tag{3}$$

Furthermore, from the sequence of the subgroups

$$SO\,(2) \times SO\,(3) \subset SO\,(4) \subset SO\,(5) \tag{4}$$

and the obvious isomorphisms $S^4 \sim SO\,(5)\,/SO\,(4)\,, S^2 \sim SO\,(3)\,/SO\,(2)$ one concludes that CP^3 may be considered as the bundle $CP^3 \to S^4$ with the fibre S^2. The fibres lie in CP^3 as projective lines $CP^1 \sim S^2$. Thus, CP^3 is locally represented as

$$CP^3 \overset{loc.}{\sim} S^4 \times S^2 \tag{5}$$

Note that the subgroup H contains solvable factor $\mathcal{R}$ (and hence H is not an unimodular), so there is no Poincaré invariant measure on M^{12}. Nevertheless from rel.(3) it follows that there is a quasi-invariant measure which becomes a genuine invariant when the Lorentz transformations are restricted to the stability subgroup of time-like vector $SO\,(5)$.

In spite of the quite intricate structure, the subgroup H admits a simple realization, namely, it can be identified with all the $SO\,(5,1)\,-$transformations multiplying the Weyl spinor λ by a complex factor

$$N_a{}^b \lambda_b = \alpha \lambda_a \quad , \quad \alpha \in C \backslash \{0\} \tag{6}$$

(all the details concerning six-dimensional spinor formalism are collected in the Appendix). This observation readily leads to the covariant parametrization of CP^3 by a complex Weyl spinor subject to the equivalence relation

$$\lambda_a \sim \alpha \lambda_a \qquad , \qquad \alpha \in C \backslash \{0\} \tag{7}$$

By construction, the Poincaré group generators act on M^{12} by the following vector fields:

$$\mathbf{P}^A = \partial^A \quad , \quad \mathbf{M}_{AB} = x_A\partial_B - x_B\partial_A - \left((\sigma_{AB})_a{}^b \lambda_b \partial^a \ + \ c.c.\right) \tag{8}$$

where $\left\{x^A\right\}$ are the Cartesian coordinates on $R^{5,1}$. It is evident that Poincaré generators commute with the projective transformations (7) generated by the vector fields

$$d = \lambda_a \partial^a \qquad , \qquad \overline{d} = \overline{\lambda}_a \overline{\partial}^a \tag{9}$$

Then the space of scalar functions on M^{12} is naturally identified with those functions $\Phi\left(x^A, \lambda_a, \overline{\lambda}_b\right)$ which satisfy the homogeneity conditions

$$d\Phi = \overline{d}\Phi = 0 \tag{10}$$

Let us consider the ring of invariant differential operators acting on the space of scalar functions on M^{12}. Such operators should commute with the Poincaré transformations (8) and the projective ones (9). It is easy to see that there are only three independent Laplace operators. They are

$$\begin{gathered} \Box = -\partial^A \partial_A \\ \triangle_1 = \lambda_a \overline{\lambda}_b \partial^b \overline{\partial}^a \quad , \quad \triangle_2 = \overline{\lambda}_a \lambda_b \partial^{ab} \partial_{cd} \overline{\partial}^c \partial^d \end{gathered} \tag{11}$$

where $\partial_{ab} = \left(\sigma^A\right)_{ab} \partial_A$. Casimir operators of the Poincaré group in representation (8) can be expressed through the Laplace operators as follows

$$\begin{gathered} C_1 = \mathbf{P}^A \mathbf{P}_A = -\Box \\ C_2 = \tfrac{1}{24}\mathbf{W}^{ABC}\mathbf{W}_{ABC} = \triangle_2 + \Box\triangle_1 \quad , \quad C_3 = \tfrac{1}{64}\mathbf{W}^A\mathbf{W}_A = \triangle_1 \triangle_2 + 2\Box\triangle_1 \end{gathered} \tag{12}$$

where $\mathbf{W}^A = \epsilon^{ABCDEF}\mathbf{M}_{BC}\mathbf{M}_{DE}\mathbf{P}_F$, $\mathbf{W}^{ABC} = \epsilon^{ABCDEF}\mathbf{M}_{DE}\mathbf{P}_F$ are Pauli-Lubanski vector and tensor respectively.

In what follows we present another covariant parametrization of M^{12} in terms of a non-zero light-like vector b^A and anti-self-dual tensor h^{ABC} constrained by the relations

$$\begin{gathered} b^A b_A = 0 \quad , \quad b^A \sim a b^A \quad , \quad h^{ABC} \sim a h^{ABC} \quad , \quad a \in R\backslash\{0\} \\ h^{ABC} = -\tfrac{1}{6}\epsilon^{ABCDEF} h_{DEF} \qquad , \qquad b_A h^{ABC} = 0 \\ h^{ABC} h_{CDE} = \tfrac{1}{4}\delta^{[A}{}_{[D} b^{B]} b_{E]} \end{gathered} \tag{13}$$

(Here the anti-self-dual tensor h^{ABC} is chosen real that is always possible in $R^{5,1}$.) As a matter of fact, the first two relations imposed on b^A define S^4 as a projective light-cone. With the use of Lorentz transformations each point on S^4 can be brought into another one parametrized by the vector $\overset{o}{b}{}^A = (1,0,0,0,0,1)$. By substituting $\overset{o}{b}{}^A$ into the fourth equation one reduces the ten components of h^{ABC} to the three independent values, for instance $h_{012}, h_{013}, h_{014}$. Then the last equation takes the form

$$\left(h_{012}\right)^2 + \left(h_{013}\right)^2 + \left(h_{014}\right)^2 = \frac{1}{4} \tag{14}$$

i.e. it defines the two-sphere S^2. In such a manner we recover the local structure of CP^3 discussed above (5). The relationship between these two parametrizations may be established explicitly with the use of the following Fierz identity:

$$\overline{\lambda}_a\lambda_b = \frac{1}{4}\overline{\lambda}\widetilde{\sigma}_A\lambda\left(\sigma^A\right)_{ab} + \frac{1}{12}\overline{\lambda}\widetilde{\sigma}_{ABC}\lambda\left(\sigma^{ABC}\right)_{ab} \tag{15}$$

Defining b^A and h^{ABC} through $\overline{\lambda}_a$,λ_a as

$$b^A = \overline{\lambda}\widetilde{\sigma}^A\lambda \qquad , \qquad h^{ABC} = i\overline{\lambda}\widetilde{\sigma}^{ABC}\lambda \tag{16}$$

one can get (13). The Poincaré generators (8) and Laplace operators (12) can straightforwardly be rewritten in terms of b^A and h^{ABC} but we omit the explicit expressions here since in what follows the spinor parametrization of CP^3 will be mainly used.

3 Action functional in the first order formalism and classical dynamics

We proceed to the derivation an action functional governing the point particle dynamics on M^{12}. The main dynamical principle underlying our construction is the requirement of identical (off-shell) conservation for the classical counterparts of three Casimir operators (12).

As a starting point, consider the phase space $T^*(R^{5,1}\times C^4)$ parametrized by the coordinates $x^A, \lambda_a, \overline{\lambda}_b$ and their conjugated momenta $p_A, \pi^a, \overline{\pi}^b$ satisfying the canonical Poisson-bracket relations

$$\left\{x^A, p_B\right\} = \delta^A{}_B \quad , \quad \left\{\lambda_a, \pi^b\right\} = \delta^b{}_a \quad , \quad \left\{\overline{\lambda}_a, \overline{\pi}^b\right\} = \delta^b{}_a \tag{17}$$

Obviously, the action of the Poincaré group on M^{12} (8) is lifted up to the canonical action on $T^*(R^{5,1}\times C^4)$. This action induces a special representation of the Poincaré group on the space of smooth functions F over the phase space, and the corresponding infinitesimal transformations can be written via the Poisson brackets as follows

$$\delta F = \left\{F, -a^A P_A + \frac{1}{2}K^{AB}J_{AB}\right\} \tag{18}$$

Here a^A and $K^{AB} = -K^{BA}$ are the parameters of translations and Lorentz transformations, respectively, and the Hamilton generators read

$$P_A = p_A \qquad , \qquad J_{AB} = x_A p_B - x_B p_A + M_{AB} \tag{19}$$

where the spinning part of Lorentz generators is given by

$$M_{AB} = -\pi\sigma_{AB}\lambda + \ c.c.$$

The phase-space counterparts of Casimir operators associated with the generators (19) can be readily obtained from (12) by making formal replacements: $\partial_A \to p_A, \partial^a \to \pi^a, \overline{\partial}^a \to \overline{\pi}^a$. The result is

$$C_1 = p^2$$
$$C_2 = p^2\left(\overline{\pi}\lambda\right)\left(\pi\overline{\lambda}\right) - \left(\overline{\pi}p\pi\right)\left(\overline{\lambda}p\lambda\right) \quad , \quad C_3 = \left(\overline{\pi}\lambda\right)\left(\pi\overline{\lambda}\right)\left(\overline{\pi}p\pi\right)\left(\overline{\lambda}p\lambda\right) \tag{20}$$

As is seen the Casimir functions C_2, C_3 are unambiguously expressed via the classical analogs of Laplace operators (11)

$$\triangle_1 = (\overline{\pi}\lambda)\left(\pi\overline{\lambda}\right) \quad , \quad \triangle_2 = (\overline{\pi}p\pi)\left(\overline{\lambda}p\lambda\right) \tag{21}$$

and, thereby, one may require the identical conservation of $\triangle_1, \triangle_2$ instead of C_2, C_3.

Let us now introduce the set of five first-class constraints, three of which are dynamical

$$\begin{gathered} T_1 = p^2 + m^2 \approx 0 \\ T_2 = \triangle_1 + \delta_1^2 \approx 0 \qquad , \qquad T_3 = \triangle_2 + m^2\delta_2^2 \end{gathered} \tag{22}$$

and the other are kinematical

$$T_4 = \pi\lambda \approx 0 \qquad , \qquad T_5 = \overline{\pi}\overline{\lambda} \approx 0 \tag{23}$$

Here parameter m is identified with the mass of the particle, while the parameters δ_1, δ_2 relate to the particle's spin. The role of kinematical constraints is to make the Hamiltonian reduction of the extended phase space to the cotangent bundle $T^*\left(M^{12}\right)$. In configuration space these constraints generate the equivalence relation (7) with respect to the Poisson brackets (17). The constraints T_1, T_2, T_3 determine the dynamical content of the model and lead to the unique choice for action functional.

¿From (22) it follows that on the constraint surface the conserved charges $\triangle_1$ and $\triangle_2$ are limited to be negative (or zero) constants. These restrictions are readily seen from the following simple reasons. Let us introduce the set of three p-transversal tensors

$$\begin{gathered} W_{ABC} = \epsilon_{ABCDEF} J^{DE} p^F \quad , \quad W_A = \epsilon_{ABCDEF} J^{BC} J^{DE} p^F \quad , \\ V_A = M_{AB} p^B \end{gathered} \tag{24}$$

Since the p is a time-like vector (22) the full contraction of each introduced tensor with itself should be non-negative. Then one may check that the following relations take place

$$\begin{gathered} W_{ABC} W^{ABC} = p^2 \triangle_1 - \triangle_2 \geq 0 \quad , \quad W_A W^A = \triangle_1 \triangle_2 \geq 0, \\ V_A V^A = -p^2 \triangle_1 - \triangle_2 \geq 0 \end{gathered} \tag{25}$$

Resolving these inequalities we come to the final relation:

$$\triangle_2 \leq m^2 \triangle_1 \leq 0 \tag{26}$$

which in turn implies that $|\delta_2| \geq |\delta_1|$. Thus, the set of constraints (22) leads to the self-consistent classical dynamics only provided that the rel.(26) holds true.

Assuming the theory to be reparametrization invariant, the Hamiltonian of the model is a linear combination of the constraints and the first-order (Hamilton) action takes the form:

$$S_H = \int d\tau \left\{ p_A\, \dot{x}^A + \pi^a\, \dot{\lambda}_a + \overline{\pi}^a\, \dot{\overline{\lambda}}_a - \sum_{i=1}^{5} e_i T^i \right\} \tag{27}$$

Here τ is the evolution parameter, e_i are the Lagrange multipliers associated to the constraints with $e_4 = \overline{e}_5$. Varying (27) one gets the following equations of motion:

$$\begin{aligned}
\dot{\lambda}_a &= e_2 \left(\overline{\pi}\lambda\right) \overline{\lambda}_a + e_3 \left(\overline{\lambda} p \lambda\right) \overline{\pi}^b p_{ba} + e_4 \lambda_a \\
\dot{\pi}^a &= -e_2 \left(\pi\overline{\lambda}\right) \overline{\pi}^a - e_3 \left(\overline{\pi} p \pi\right) \overline{\lambda}_b p^{ba} - e_4 \pi^a \\
\dot{x}^A &= 2 e_1 p^A + e_3 \left\{ \left(\overline{\pi}\sigma^A \pi\right) \left(\overline{\lambda} p \lambda\right) + \left(\overline{\pi} p \pi\right) \left(\overline{\lambda}\tilde{\sigma}^A \lambda\right) \right\} \\
\dot{p}_A &= 0
\end{aligned} \tag{28}$$

Despite the quite nonlinear structure, the equations are found to be completely integrable with arbitrary Lagrange multipliers. This fact is not surprising as the model, by construction, describes a free relativistic particle possessing a sufficient number of symmetries.

In the spinning sector the corresponding solution looks like:

$$\begin{aligned}
\lambda_a &= e^{E_4} \cos\left(m^2 E_3 \delta_2\right) \left(\cos\left(E_2\delta_1\right) \lambda^0_a + \frac{\sin\left(E_2\delta_1\right)}{\delta_1} \left(\overline{\pi}_0 \lambda^0\right) \overline{\lambda}^0_a \right) + \\
&+ e^{E_4} \frac{\left(\overline{\lambda}^0 p \lambda^0\right)}{m^2} \frac{\sin\left(m^2 E_3 \delta_2\right)}{\delta_2} p_{ab} \left(\frac{\sin\left(E_2\delta_1\right)}{\delta_1} \left(\overline{\pi}_0 \lambda^0\right) \pi^b_0 - \cos\left(E_2\delta_1\right) \overline{\pi}^b_0 \right) \\
\pi^a &= e^{-E_4} \cos\left(m^2 E_3 \delta_2\right) \left(\cos\left(E_2\delta_1\right) \pi^a_0 - \frac{\sin\left(E_2\delta_1\right)}{\delta_1} \left(\pi_0 \overline{\lambda}^0\right) \overline{\pi}^a_0 \right) + \\
&+ e^{-E_4} \frac{\left(\overline{\pi}_0 p \pi_0\right)}{m^2} \frac{\sin\left(m^2 E_3 \delta_2\right)}{\delta_2} p^{ab} \left(\frac{\sin\left(E_2\delta_1\right)}{\delta_1} \left(\pi_0 \overline{\lambda}^0\right) \lambda^0_b + \cos\left(E_2\delta_1\right) \overline{\lambda}^0_b \right)
\end{aligned} \tag{29}$$

and for the space-time evolution one gets

$$\begin{aligned}
p^A &= p^A_0 \\
x^A(\tau) &= x^A_0 + 2\left(E_1 + E_3 \delta^2_2\right) p^A_0 - m^{-2} V^A(\tau) \\
V^A(\tau) &= V^A_1 \cos\left(2m^2 E_3 \delta_2\right) + V^A_2 \sin\left(2m^2 E_3 \delta_2\right)
\end{aligned} \tag{30}$$

Here $E_i(\tau) = \int\limits_0^\tau d\tau e_i(\tau)$, vector V^A is defined as in (24) and the initial data $\lambda^0_a = \lambda_a(0)$, $\pi^a_0 = \pi^a(0)$, p^A_0 are assumed to be chosen on the surface of constraints (22), (23).

Let us briefly discuss the obtained solution. First of all, one may resolve the kinematical constraints (23) by imposing the gauge fixing conditions of the form

$$e_4 = e_5 = 0 \qquad , \qquad \lambda_0 = 1\,, \tag{31}$$

so that $\lambda_i, i = 1, 2, 3$ can be treated as the local coordinates on CP^3. Then from (29), (30) we see that the motion of the point particle on M^{12} is completely determined by an

independent evolution of the three Lagrange multipliers e_1, e_2, e_3. The presence of two additional gauge invariances in comparison with spinless particle case causes the conventional notion of particle world line, as the geometrical set of points, to fail. Instead, one has to consider the class of gauge equivalent trajectories on M^{12} which, in the case under consideration, is identified with three-dimensional surface, parametrized by e_1, e_2, e_3 . The space-time projection of this surface is represented by the two-dimensional tube of radius $\rho = \sqrt{\delta_2^2 - \delta_1^2}$ along the particle's momenta p as is seen from the explicit expression (30). This fact becomes more clear in the rest reference system $\overset{\circ}{p}_A = (m, \vec{0})$ after identifying of the evolution parameter τ with the physical time by the law

$$x^0 = c\tau \tag{32}$$

Then eq. (30) reduces to

$$\vec{x}\,(\tau) = m^{-2}\,\vec{V}\,(\tau) = \vec{V}_1 \cos\left(2m^2 E_3 \delta_2\right) + \vec{V}_2 \sin\left(2m^2 E_3 \delta_2\right) \tag{33}$$

where, in accordance with (25) $\vec{V}^{\,2} = m^2\,(\delta_2^2 - \delta_1^2)$ and hence

$$\vec{V}_1^{\,2} = \vec{V}_2^{\,2} = \delta_2^2 - \delta_1^2\ ,\ (\vec{V}_1, \vec{V}_2) = 0 \tag{34}$$

The rest gauge arbitrariness, related to the Lagrange multiplier e_3, causes that, in each moment of time, the space-time projection of the motion is represented by a circle of radius ρ. This means that after accounting spin, the relativistic particle ceases to be localized in a certain point of Minkowski space but represents a string-like configuration contracting to the point only provided that $\delta_1 = \delta_2$.

Finally, let us discuss the structure of the physical observables of the theory. Each physical observable A being a gauge-invariant function on the phase space should meet the requirements:

$$\{A, T_i\} = 0 \qquad , \qquad i = 1, .., 5 \tag{35}$$

Due to the obvious Poincaré invariance of the constraint surface, the generators (19) automatically satisfy (35) and thereby they are the observables. On the other hand, it is easy to compute that the dimension of the physical phase space of the theory is equal to 18. Thus the physical subspace may covariantly be parametrized by 21 Poincaré generator subject to 3 conditions (20), and as a result, any physical observable proves to be a function of the generators (19) modulo constraints. So a general solution of (35) reads

$$A = A\,(J_{AB}, P_C) + \sum_{i=1}^{5} \alpha_i T_i \tag{36}$$

α_i, being an arbitrary function of phase space variables.

In fact, this implies that the physical phase space of the model is embedded in the linear space of the Poincaré algebra through the constraints (22) and therefore coincides with some coadjoint orbit $\mathcal{O}$ of the Poincaré group.

4 Second-order formalism

In order to obtain a second-order formulation for the model one may proceed in the standard manner by eliminating the momenta $p_A, \pi^a, \overline{\pi}^a$ and the Lagrange multipliers e_i from the Hamiltonian action (27) resolving equations of motion:

$$\frac{\delta S}{\delta p_A} = \frac{\delta S}{\delta \pi^a} = \frac{\delta S}{\delta \overline{\pi}^a} = \frac{\delta S}{\delta e_i} = 0 \tag{37}$$

with respect to the momenta and the multipliers. The corresponding Lagrangian action will be invariant under global Poincaré transformation and will possess five gauge symmetries associated with first-class constraints (22). The presence of kinematical ones will result in the invariance of Lagrangian action under the local λ-rescalings: $\lambda_a \to \alpha\lambda_a$. At the same time, by construction, among the gauge transformations related to the dynamical constraints will necessarily be the one corresponding to reparametrizations of the particle world-line $\tau \to \tau'(\tau)$.

It turns out, however, that the straightforward resolution of eqs. (37) is rather cumbersome. Fortunately, in the case in hand there is another way to recover the covariant second-order formulation exploiting the symmetry properties of the model. Namely, we can start with the most general Poincaré and reparametrization invariant ansatz for the Lagrange action and specify it, by requiring the model to be equivalent to that described by the constraints (22).

As a first step we classify all the Poincaré invariants of the world-line being functions over the tangent bundle TM^{12}. One may easily verify that there are only three expressions possessing these properties

$$\dot{x}^2 \qquad , \qquad \xi = \frac{(\dot{\lambda}\dot{x}\,\lambda)(\dot{\overline{\lambda}}\dot{x}\,\overline{\lambda})}{\dot{x}^2\left(\overline{\lambda}\,\dot{x}\,\lambda\right)^2} \qquad , \qquad \eta = \frac{\epsilon^{abcd}\,\dot{\lambda}_a\,\overline{\lambda}_b\,\dot{\overline{\lambda}}_c\,\lambda_d}{\left(\overline{\lambda}\,\dot{x}\,\lambda\right)^2} \tag{38}$$

Notice that ξ and η are invariant under reparametrizations as well as under the local λ-rescalings (7), so the kinematical constraints (23) are automatically accounted

Then the most general Poincaré and reparametrization invariant Lagrangian on M^{12} reads

$$\mathcal{L} = \sqrt{-\,\dot{x}^2\;F\,(\xi,\eta)} \tag{39}$$

where F is an arbitrary function.

The particular form of the function F entering (39) may be found from the requirement that the Lagrangian is to lead to the Hamilton constraints (22). The substitution of the canonical momenta

$$p_A = \frac{\partial\mathcal{L}}{\partial\,\dot{x}^A} \qquad , \qquad \pi^a = \frac{\partial\mathcal{L}}{\partial\,\dot{\lambda}_a} \qquad , \qquad \overline{\pi}^a = \frac{\partial\mathcal{L}}{\partial\,\dot{\overline{\lambda}}_a} \tag{40}$$

to the dynamical constraints T_1 and T_2 gives the following equations

$$\begin{gathered}\frac{\partial\mathcal{L}}{\partial\,\dot{x}^A}\frac{\partial\mathcal{L}}{\partial\,\dot{x}_A} + m^2 = 0 \Rightarrow \\ \Rightarrow F^2 + \xi\,(\xi+\eta)\left(\frac{\partial F}{\partial\xi}\right)^2 - 2\xi\frac{\partial F}{\partial\xi} - 2\eta\frac{\partial F}{\partial\eta} + 2\xi\eta\frac{\partial F}{\partial\xi}\frac{\partial F}{\partial\eta} - m^2 F = 0\end{gathered} \tag{41}$$

and

$$\frac{\partial \mathcal{L}}{\partial \dot{\overline{\lambda}}_a}\lambda_a \frac{\partial \mathcal{L}}{\partial \dot{\lambda}_b}\overline{\lambda}_b + \delta_1^2 = 0 \Rightarrow \left(\frac{\partial F}{\partial \xi}\right)^2 + \delta_1^2 F = 0 \tag{42}$$

The integration of these equations results with

$$F = \left(2\delta_1\sqrt{-\xi} + \sqrt{m^2 - 4\delta_1^2\eta + 4A\sqrt{\eta}}\right)^2 \quad , \tag{43}$$

A being arbitrary constant of integration. The account of the rest constraint T_3 does not contradict the previous equations, but determines the value of A as

$$A = m\sqrt{\delta_2^2 - \delta_1^2} \tag{44}$$

Putting altogether, we come with the Lagrangian

$$\mathcal{L} = \sqrt{-\dot{x}^2\left(m^2 - 4\delta_1^2\frac{\epsilon^{abcd}\,\dot{\lambda}_a\,\overline{\lambda}_b\,\dot{\overline{\lambda}}_c\,\lambda_d}{\left(\overline{\lambda}\,\dot{x}\,\lambda\right)^2} + 4m\sqrt{\left(\delta_2^2 - \delta_1^2\right)\frac{\epsilon^{abcd}\,\dot{\lambda}_a\,\overline{\lambda}_b\,\dot{\overline{\lambda}}_c\,\lambda_d}{\left(\overline{\lambda}\,\dot{x}\,\lambda\right)^2}}\right)} + \tag{45}$$

$$+2\delta_1\left|\frac{\dot{\overline{\lambda}}\dot{x}\,\lambda}{\overline{\lambda}\,\dot{x}\,\lambda}\right|$$

It should be stressed that the parameters δ_1 and δ_2 entering the Lagrangian are dimensional and cannot be made dimensionless by redefinitions involving only the mass of the particle and the speed of light c. Whereas, using the Planck constant we may set

$$\delta_1 = \frac{\hbar}{c}\kappa_1 \qquad , \qquad \delta_2 = \frac{\hbar}{c}\kappa_2 \tag{46}$$

where κ_1 and κ_2 are already arbitrary real numbers satisfying the inequality $|\kappa_1| \le |\kappa_2|$. Turning back to the question of particle motion (see (30) and below) we also conclude that the radius ρ of the tube, representing the particle propagation in Minkowski space is proportional to $\hbar$. So, this "non-local" behavior of the particle is caused by spin which manifests itself as a pure quantum effect disappearing in the classical limit $\hbar \to 0$.

As is seen, for a given non-zero, spin the Lagrangian (45) has a complicated structure involving radicals and, hence, the reality condition for $\mathcal{L}$ requires special consideration. Similar to the spinless case, the space-time causality implies that

$$\dot{x}^2 < 0 \qquad , \qquad \dot{x}^0 > 0 \tag{47}$$

Then expression (45) is obviously well-defined only provided that

$$\begin{gathered} \eta \ge 0 \\ m^2 - 4\delta_1^2\eta + 4m\sqrt{\left(\delta_2^2 - \delta_1^2\right)\eta} \ge 0 \end{gathered} \tag{48}$$

As will be seen below the first inequality is always fulfilled, while the second condition is equivalent to

$$0 \le \eta \le \frac{m^2}{4\delta_1^4}\left(\delta_2 + \sqrt{\delta_2^2 - \delta_1^2}\right)^2 \tag{49}$$

Together, eqs. (47), (49) may be understood as the full set of causality conditions for the model of massive spinning particle.

Passing to the vector parametrization of the configuration space in terms of b_A and h_{ABC} the basic invariants η and ξ take the form

$$\begin{aligned}\xi &= -\frac{4\,\dot{x}_A \dot{h}^{ABC}\dot{h}_{BCD}\dot{x}^D + \dot{x}^2\dot{b}^2 - 4\left(\dot{x}\dot{b}\right)^2}{16\,\dot{x}^2\,(\dot{x}\,b)^2}\\ \eta &= \frac{\dot{b}^2}{4\,(\dot{x}\,b)^2}\end{aligned} \tag{50}$$

and the corresponding Lagrangian reads

$$\begin{aligned}\mathcal{L} = &\sqrt{-\,\dot{x}^2\left(m^2 - \delta_1^2\frac{\dot{b}^2}{(\dot{x}\,b)^2} + 2m\sqrt{(\delta_2^2 - \delta_1^2)\,\frac{\dot{b}^2}{(\dot{x}\,b)^2}}\right)} +\\ &+\delta_1\sqrt{\frac{4\,\dot{x}_A \dot{h}^{ABC}\dot{h}_{BCD}\dot{x}^D + \dot{x}^2\dot{b}^2 - 4\left(\dot{x}\dot{b}\right)^2}{4\,(\dot{x}\,b)^2}}\end{aligned} \tag{51}$$

where the holonomic constraints (13) are assumed to hold. In view of (50) the condition (48) becomes evident since $\dot{b}^A$ is orthogonal to the light-like vector b^A and thereby is space-(or light-) like. Recalling that the vector b^A parametrizes S^4, condition (49) forbids the particle to move with arbitrary large velocity not only in Minkowski space but also on the sphere S^4.

Classically the parameters δ_1 and δ_2 can be chosen to be arbitrary numbers subject only to the restriction $|\delta_1| \le |\delta_2|$. There are, however, two special cases: $\delta_1 = \delta_2 = 0$ and $\delta_1 = 0$ when the Lagrangian (51) is considerably simplified. The former option is of no interest as it corresponds to the case of spinless massive particle, while the latter leads to the following Lagrangian

$$\mathcal{L} = \sqrt{-\,\dot{x}^2\left(m^2 + 2m\delta_2\sqrt{\frac{\dot{b}^{\,2}}{(\dot{x}\,b)^2}}\right)} \tag{52}$$

which is the direct six-dimensional generalization of the (m, s)-particle model proposed earlier [4] for $D = 4$. The configuration space of the model (52) is represented by the direct product of Minkowski space $R^{5,1}$ and four-dimensional sphere S^4 parametrized by the light-like vector b^A. It is easy to see that the reduced model cannot describe arbitrary spins, since the third Casimir operator (12), being constructed from the Poincaré generators acting on $R^{5,1}\times S^4$, vanishes identically. As will be seen below the quantization of this case leads to the irreducible representations of the Poincaré group realized on totally symmetric tensor fields on Minkowski space.

5 Generalization to the curved background

So far we discussed the model of spinning particle living on the flat space-time. In this section, we will try to generalize it to the case of curved background. For these ends one can replace the configuration space M^{12} by $\mathcal{M}^6 \times CP^3$ where $\mathcal{M}^6$ is a curved space-time. Now the action functional should be generalized to remain invariant under both general coordinate transformations on $\mathcal{M}^6$ and local Lorentz transformations on CP^3. Let $e_m{}^A$ and ω_{mAB} be the vielbein and the torsion free spin connection respectively. The minimal covariantization of the Lagrangian (45) gives

$$\mathcal{L} = \sqrt{-\dot{x}^2\left(m^2 - 4\delta_1^2 \frac{\epsilon^{abcd}\, \overset{\bullet}{\lambda}_a\, \overline{\lambda}_b\, \overset{\bullet}{\overline{\lambda}}_c\, \lambda_d}{\left(\dot{x}^m\, e_m{}^A \left(\overline{\lambda}\sigma_A\lambda\right)\right)^2} + 4m\sqrt{\left(\delta_2^2 - \delta_1^2\right)\frac{\epsilon^{abcd}\, \overset{\bullet}{\lambda}_a\, \overline{\lambda}_b\, \overset{\bullet}{\overline{\lambda}}_c\, \lambda_d}{\left(\dot{x}^m\, e_m{}^A \left(\overline{\lambda}\sigma_A\lambda\right)\right)^2}}\right) +} \tag{53}$$

$$+2\delta_1 \left| \frac{\dot{x}^m\, e_m{}^A \left(\overset{\bullet}{\overline{\lambda}}\, \sigma_A\lambda\right)}{\dot{x}^m\, e_m{}^A \left(\overline{\lambda}\sigma_A\lambda\right)} \right|$$

where

$$\overset{\bullet}{\lambda}_a = \dot{\lambda}_a - \frac{1}{2}\, \dot{x}^m\, \omega_{mAB}(\sigma^{AB})_a{}^b\lambda_b \tag{54}$$

is the Lorentz covariant derivative along the particle's world line.

Proceeding to the Hamilton formalism one gets the set of five constraints $T_i^{'}$, $i = 1...5$ which may be obtained from T_i (22,23) by replacing $p_A \to \Pi_A$, where

$$\Pi_A = e_A{}^m \left(p_m + \frac{1}{2}\omega_{mCD}M^{CD}\right) \tag{55}$$

Here $e_A{}^m$ is the inverse vielbein and M^{CD} is the spinning part of Lorentz generators (19). The generalized momentum Π_A satisfies the following Poisson brackets relation:

$$\{\Pi_A, \Pi_B\} = \frac{1}{2}R_{ABCD}M^{CD} \tag{56}$$

R_{ABCD} being the curvature tensor of $\mathcal{M}^6$. Now it is easy to find that

$$\begin{gathered} \left\{T_1^{'}, T_3^{'}\right\} = R_{ABCD}q^A\Pi^B M^{CD} \\ q^A = (\overline{\lambda}\sigma^A\lambda)(\overline{\pi}\Pi\pi) + (\overline{\lambda}\Pi\lambda)(\overline{\pi}\sigma^A\pi) \end{gathered} \tag{57}$$

The other Poisson brackets of the constraints are equal to zero. So, in general, the constraints $T_1^{'}, T_3^{'}$ are of the second class, which implies that switching on an interaction destroys the first class constraints algebra and, hence, gives rise to unphysical degrees of freedom in the theory. What is more, the Lagrangian (53) is explicitly invariant under reparametrizations of the particle's world line, while the gauge transformations, associated with the remaining first class constraints $T_2^{'}, T_4^{'}, T_5^{'}$, do not generate the full reparametrizations of the theory (the space-time coordinates x^m on $\mathcal{M}^6$ remain intact). The last fact

indicates that the equations of motion derived from (53) are contradictory. Thus the interaction with external gravitational field is self-consistent only provided that r.h.s. of (57) vanishes. This requirement leads to some restrictions on curvature tensor. Namely, with the use of the identity $M^{AB}q_B \approx 0$ one may find that (57) is equal to zero if and only if R_{ABCD} has the form

$$R_{ABCD} = \frac{R}{30}\left(\eta_{AC}\eta_{BD} - \eta_{AD}\eta_{BC}\right) \tag{58}$$

where R is a constant (the scalar curvature of the manifold $\mathcal{M}^6$). So the minimal coupling to gravity is self-consistent only provided that $\mathcal{M}^6$ is the space of constant curvature.

Concluding this section let us also remark that the Lagrangian (53) may be obtained using the group theoretical principles outlined in the introduction. To this end one should replace the Poincaré group by $SO\,(5,2)$ or $SO\,(6,1)$ depending on $R<0$ or $R>0$. (Cf. see [4])

6 Quantization

In Sect. 3 we have seen that the model is completely characterized, at the classical level, by the algebra of observables associated with the phase space generators of the Poincaré group. We have shown that the observables $\mathcal{A} = (P_A, J_{AB})$ constitute the basis, so that any gauge invariant value of the theory can be expressed via the elements of $\mathcal{A}$.

To quantize this classical system means to construct an irreducible unitary representation

$$r : \mathcal{A} \rightarrow End\ \mathcal{H} \tag{59}$$

of the Lie algebra $\mathcal{A}$ in the algebra $End\ \mathcal{H}$ of linear self-adjoint operators in a Hilbert space where the physical subspace $\mathcal{H}$ is identified with the kernel of the first-class constraint operators. Here by a Lie algebra representation r we mean a linear mapping from $\mathcal{A}$ into $End\ \mathcal{H}$ such that

$$r(\{f,g\}) = -i[r(f), r(g)] \qquad , \qquad \forall\ f,g \in \mathcal{A} \tag{60}$$

where $[r(f), r(g)]$ is the usual commutator of Hermitian operators $r(f)$, $r(g)$. Unitarity means that the canonical transformations of the model's phase space generated by observables from $\mathcal{A}$ should correspond to unitary transformations on $\mathcal{H}$. Besides that we should supply the algebra $\mathcal{A}$ by the central element 1 and normalize r by the condition

$$r(1) = id \tag{61}$$

i.e. the constant function equal to 1 corresponds under r to the identity operator on $\mathcal{H}$.

Now it is seen *that the quantization of the model is reduced to the construction of the unitary irreducible representation of the Poincaré group with the given quantum numbers fixed by the constraints* (22, 23).

Within the framework of the covariant operatorial quantization the Hilbert space of physical states $\mathcal{H}$ is embedded into the space of smooth scalar functions on $R^{5,1} \times C^4$ and the phase space variables $x^A, p_A, \lambda_a, \pi^a$ are considered to be Hermitian operators subject to the canonical commutation relations.

In the ordinary coordinate representation

$$p_A \to -i\partial_A \qquad , \qquad \pi^a \to -i\partial^a \qquad , \qquad \overline{\pi}^a \to -i\overline{\partial}^a \tag{62}$$

the Hermitian generators of the Poincaré group (observables) take the form

$$\mathbf{P}_A = -i\partial_A \quad , \qquad \mathbf{M}_{AB} = -i\left(x_A\partial_B - x_B\partial_A + (\sigma_{AB})_a{}^b\left(\lambda_b\partial^a + \overline{\lambda}_b\overline{\partial}^a\right)\right) \tag{63}$$

By contrast, the quantization of the first-class constraints is not so unambiguous. As is seen from the explicit expressions (22, 23) there is an inherent ambiguity in the ordering of operators $\widehat{\lambda}_a, \widehat{\pi}^b$ and $\widehat{\overline{\lambda}}_a, \widehat{\overline{\pi}}^b$. Luckily as one may verify, the different ordering prescription for the above operators results only in renormalization of the parameters δ_1^2, δ_2^2 and modification of the kinematical constraints by some constants n and m. Thus, in general, (after omitting inessential multipliers) the quantum operators for the first-class constraints may be written as

$$\begin{gathered}\widehat{T}_1 = \Box - m^2 \quad , \quad \widehat{T}_2 = \triangle_1 - \delta_1^{\prime 2} \quad , \quad \widehat{T}_3 = \triangle_2 - \delta_2^{\prime 2} \\ \widehat{T}_4 = d - n \qquad , \qquad \widehat{T}_5 = \overline{d} - m\end{gathered} \tag{64}$$

where the operators in the r.h.s. of relations are defined as in (9), (11), and $\delta_1^{\prime 2}, \delta_2^{\prime 2}$ are renormalized parameters δ_1^2, δ_2^2.

The subspace of physical states $\mathcal{H}$ is then extracted by conditions

$$\widehat{T}_i\left|\Phi_{phys}\right\rangle = 0 \qquad , \qquad i = 1, ..., 5 \tag{65}$$

The imposition of the kinematical constraints yields that the physical wave functions are homogeneous in λ and $\overline{\lambda}$ of bedegree (n, m) i.e.

$$\Phi\left(x, \alpha\lambda, \overline{\alpha}\overline{\lambda}\right) = \alpha^n\overline{\alpha}^m\Phi\left(x, \lambda, \overline{\lambda}\right) \tag{66}$$

From the standpoint of the intrinsic M^{12} geometry these functions can be interpreted as the special tensor fields being the scalars on Minkowski space $R^{5,1}$ and, simultaneously the densities of weight (n, m) with respect to the holomorphic transformations of CP^3. Requiring the fields (66) to be unambiguously defined on the manifold, the parameters n and m should be restricted to be integer.

Let us consider the space ${}^{\uparrow}\mathcal{H}^{[0]}(M^{12}, m)$ of massive positive frequency fields of the type (0,0) (i.e. the scalar fields on M^{12}). Such fields satisfy the mass-shell condition

$$\left(\Box - m^2\right)\Phi\left(x, \lambda, \overline{\lambda}\right) = 0 \tag{67}$$

and possess the Fourier decomposition

$$\begin{gathered}\Phi\left(x, \lambda, \overline{\lambda}\right) = \int \frac{d\,\overrightarrow{p}}{p_0} e^{i(p,x)}\Phi\left(p, \lambda, \overline{\lambda}\right) \\ p^2 + m^2 = 0 \quad , \quad p_0 > 0\end{gathered} \tag{68}$$

The space ${}^{\uparrow}\mathcal{H}^{[0]}(M^{12},m)$ may be endowed with the Poincaré-invariant and positive-definite inner product defined by the rule

$$\langle \Phi_1 | \Phi_2 \rangle = \int \frac{d\,\overrightarrow{p}}{p_0} \int\limits_{CP^3} \overline{\omega} \wedge \omega \overline{\Phi}_1 \Phi_2 \tag{69}$$

where the three-form ω is given by

$$\omega = \frac{\epsilon^{abcd} \lambda_a d\lambda_b \wedge d\lambda_c \wedge d\lambda_d}{\left(\overline{\lambda} p \lambda\right)^2} \tag{70}$$

Then ${}^{\uparrow}\mathcal{H}^{[0]}(M^{12},m)$ becomes the Hilbert space and, as a result, the Poincaré representation acting on this space by the generators (63) is unitary. This representation can be readily decomposed into the direct sum of irreducible ones by means of Laplace operators $\triangle_1$ and $\triangle_2$. Namely, the subspace of irreducible representation proves to be the eigenspace for both Laplace operators. This implies the following

$${}^{\uparrow}\mathcal{H}^{[0]}(M^{12},m) = \bigoplus_{\substack{s_1,s_2=0,1,2,\ldots \\ s_1 \geq s_2}} {}^{\uparrow}\mathcal{H}_{s_1,s_2}(M^{12},m) \tag{71}$$

and the spectrum of Laplace operators is given by the eigenvalues

$$\begin{gathered} \delta_1'^2 = s_2 \left(s_2 + 1\right) \;, \quad \delta_2'^2 = m^2 s_1 \left(s_1 + 3\right) \\ s_1 \geq s_2 \qquad , \qquad s_1, s_2 = 0, 1, 2, \ldots \end{gathered} \tag{72}$$

Consequently, the subspace of physical states satisfying the quantum conditions (65) is exactly ${}^{\uparrow}\mathcal{H}_{s_1,s_2}(M^{12},m)$. The explicit expression for an arbitrary field from ${}^{\uparrow}\mathcal{H}_{s_1,s_2}(M^{12},m)$ reads

$$\Phi\left(p,\lambda,\overline{\lambda}\right) = \Phi\left(p\right)^{a_1 \ldots a_{s_1+s_2} b_1 \ldots b_{s_1-s_2}} \frac{\lambda_{a_1} \ldots \lambda_{a_{s_1}} \overline{\lambda}_{a_{s_1+1}} .. \overline{\lambda}_{a_{s_1+s_2}} \overline{\lambda}_{b_1} \ldots \overline{\lambda}_{b_{s_1-s_2}}}{\left(\overline{\lambda} p \lambda\right)^{s_1}} \tag{73}$$

Here the spin-tensor $\Phi\left(p\right)^{a_1 \ldots a_{s_1+s_2} b_1 \ldots b_{s_1-s_2}}$ is considered to be the p-transversal

$$p_{a_1 b_1} \Phi\left(p\right)^{a_1 \ldots a_{s_1+s_2} b_1 \ldots b_{s_1-s_2}} = 0 \tag{74}$$

(for $s_1 \neq s_2$) and its symmetry properties are described by the following Young tableaux:

a_1	$\cdot\;\cdot\;\cdot$	a_n		$\cdot\;\cdot\;\cdot$	a_m	$n = s_1 - s_2$
b_1	$\cdot\;\cdot\;\cdot$	b_n				$m = s_1 + s_2$

The field $\Phi\left(p\right)^{a_1 \ldots a_{s_1+s_2} b_1 \ldots b_{s_1-s_2}}$ can be identified with the Fourier transform of spin-tensor field on Minkowski space $R^{5,1}$. Together, mass-shell condition

$$\left(p^2 + m^2\right) \Phi\left(p\right)^{a_1 \ldots a_{s_1+s_2} b_1 \ldots b_{s_1-s_2}} = 0 \tag{75}$$

and relation (74) constitute the full set of relativistic wave equations for the mass-m, spin-(s_1, s_2) field in six dimensions. Thus the massive scalar field on M^{12} generates fields of arbitrary integer spins on Minkowski space.

In order to describe the half-integer spin representations of Poincaré group consider the space ${}^{\uparrow}\mathcal{H}^{[1/2]}\left(M^{12},m\right)$ of massive positive frequency fields with tensor type (1,0). These fields possess the Fourier decomposition and may be endowed with the following Hermitian inner product

$$\langle\Phi_1\,|\Phi_2\rangle_{1/2} = \int\frac{d\overrightarrow{p}}{p_0}\int\limits_{CP^3}\overline{\omega}\wedge\omega\left(\overline{\lambda}p\lambda\right)^{-1}\overline{\Phi}_1\Phi_2 \tag{76}$$

Then the decomposition of the space ${}^{\uparrow}\mathcal{H}^{[1/2]}\left(M^{12},m\right)$ with respect to both Laplace operators reads

$${}^{\uparrow}\mathcal{H}^{[1/2]}(M^{12},m) = \bigoplus_{\substack{s_1,s_2=1/2,3/2,...\\ s_1\geq s_2}}{}^{\uparrow}\mathcal{H}_{s_1,s_2}(M^{12},m) \tag{77}$$

where invariant subspaces ${}^{\uparrow}\mathcal{H}_{s_1,s_2}(M^{12},m)$ are the eigenspaces of $\triangle_1$ and $\triangle_2$ with eigenvalues

$$\begin{gathered}\delta_1'^2 = (s_2-1/2)\,(s_2+3/2)\quad,\qquad \delta_2'^2 = (s_1-1/2)\,(s_1+7/2)\\ s_1,s_2 = 1/2,3/2,...\quad,\qquad s_1\geq s_2\end{gathered} \tag{78}$$

The explicit structure of an arbitrary field from ${}^{\uparrow}\mathcal{H}_{s_1,s_2}(M^{12},m)$ is

$$\Phi\left(p,\lambda,\overline{\lambda}\right) = \Phi\left(p\right)^{a_1...a_{s_1+s_2}b_1...b_{s_1-s_2}}\frac{\lambda_{a_1}...\lambda_{a_{s_1}}\overline{\lambda}_{a_{s_1+1}}..\overline{\lambda}_{a_{s_1+s_2}}\overline{\lambda}_{b_1}...\overline{\lambda}_{b_{s_1-s_2}}}{\left(\overline{\lambda}p\lambda\right)^{s_1}} \tag{79}$$

where $\Phi\left(p\right)^{a_1...a_{s_1+s_2}b_1...b_{s_1-s_2}}$ is the p-transversal tensor

$$p_{a_1b_1}\Phi\left(p\right)^{a_1...a_{s_1+s_2}b_1...b_{s_1-s_2}} = 0 \tag{80}$$

(for $s_1\neq s_2$) and its symmetry properties are described by the above written Young tableaux. Consequently, from (77), (78) it follows that the massive type (1,0) field on M^{12} generates fields of arbitrary half-integer spins on Minkowski space.

It is instructive to rewrite the inner product for two fields from ${}^{\uparrow}\mathcal{H}_{s_1,s_2}(M^{12},m)$ in terms of spin-tensors $\Phi\left(p\right)^{a_1...a_{s_1+s_2}b_1...b_{s_1-s_2}}$. The integration over spinning variables may be performed with the use of the basic integral

$$\int\limits_{CP^3}\overline{\omega}\wedge\omega = \frac{48i\pi^3}{\left(p^2\right)^2} \tag{81}$$

and the result is

$$\langle\Phi_1\,|\Phi_2\rangle = N\int\frac{d\,\overrightarrow{p}}{p_0}\overline{\Phi}_1\left(p\right)^{a_1...a_{2s_1}}\Phi_2\left(p\right)_{a_1...a_{2s_1}} \tag{82}$$

where

$$\begin{gathered}\Phi_2\left(p\right)_{a_1...a_mb_1...b_n} =\\ = \epsilon_{a_1b_1c_1d_1}...\epsilon_{a_nb_nc_nd_n}p_{a_{n+1}c_{n+1}}...p_{a_mc_m}\Phi_2\left(p\right)^{c_1...c_md_1...d_n}\end{gathered} \tag{83}$$

and N is some normalization constant depending on s_1 and s_2.

7 Conclusion

In this paper we have suggested the model for a massive spinning particle in six-dimensional Minkowski space as a mechanical system with configuration space $M^{12} = R^{5,1} \times CP^3$. The Lagrangian of the model is unambiguously constructed from the M^{12} world line invariants when the identical conservation is required for the classical counterparts of Casimir operators. As a result, the theory is characterized by three genuine gauge symmetries.

The model turns out to be completely solvable as it must, if it is a free relativistic particle. The projection of the class of gauge equivalent trajectories from $M^{12} = R^{5,1} \times CP^3$ onto $R^{5,1}$ represents the two-dimensional cylinder surface of radius $\rho \sim \hbar$ with generatings parallel to the particle's momenta.

Canonical quantization of the model naturally leads to the unitary irreducible representation of Poincaré group. The requirement of the existence of smooth solutions to the equations for the physical wave functions results in quantization of the parameters entering Lagrangian or, that is the same, in quantization of particle's spin.

It should be noted that switching on an interaction of the particle to the inhomogeneous external field, one destroys the first class constraint algebra of the model and the theory, thereby, becomes inconsistent, whereas the homogeneous background is admissible. The physical cause underlying this inconsistency is probably that the local nature of the inhomogeneous field may contradict to the nonlocal behavior of the particle dynamical histories. A possible method to overcome the obstruction to the interaction is to involve the Wess-Zumino like invariant omitted in the action (45). It has the form

$$\Gamma = \rho \frac{(\overline{\lambda}_a \, \dot{x}^{ab} \dot{\lambda}_b)}{\left(\overline{\lambda}_a \, \dot{x}^{ab} \, \lambda_b\right)} + \overline{\rho} \frac{(\dot{\overline{\lambda}}_a \dot{x}^{ab} \, \lambda_b)}{\left(\overline{\lambda}_a \, \dot{x}^{ab} \, \lambda_b\right)}$$

As is easy to see, Γ is invariant under the λ-rescalings up to a total derivative only. This fact, however, does not prevent to say about the particle's dynamics on M^{12}. The similar trick solves the problem of interaction in the case of $d = 4$ spinning particle [5].

8 Acknowledgments

The authors would like to thank I. A. Batalin, I. V. Gorbunov, S. M. Kuzenko, A. Yu. Segal and M. A. Vasiliev for useful discussions on various topics related to the present research. The work is partially supported by the European Union Commission under the grant INTAS 93-2058. S. L. L. is supported in part by the grant RBRF 96-01-00482.

9 Appendix. Half-spinorial formalism in six dimensions

Our notations are as follows: capital Latin letters are used for Minkowski space indices and small Latin letters for spinor ones. The metric is chosen in the form: $\eta_{AB} = diag(-, +, ..., +)$. The Clifford algebra of 8×8 Dirac matrices Γ_A reads: $\{\Gamma_A, \Gamma_B\} =$

$-2\eta_{AB}$. The suitable representation for Γ_A is

$$\Gamma_A = \begin{pmatrix} 0 & (\sigma_A)_{a\dot{a}} \\ (\widetilde{\sigma}_A)^{\dot{a}a} & 0 \end{pmatrix} \quad , \qquad \begin{array}{c} \sigma_A = \{1, \gamma_0, i\gamma_1, i\gamma_2, i\gamma_3, \gamma_5\} \\ \widetilde{\sigma}_A = \{1, -\gamma_0, -i\gamma_1, -i\gamma_2, -i\gamma_3, -\gamma_5\} \end{array} \tag{84}$$

where $\gamma_i, i = 0, 1, 2, 3, 5$ are the ordinary Dirac matrices in four dimensions. The charge conjugation matrix is defined as

$$C = \Gamma_2\Gamma_4 = \begin{pmatrix} I & 0 \\ 0 & \widetilde{I} \end{pmatrix} \quad , \quad I = \widetilde{I} = \left(\begin{array}{cc|cc} 0 & 1 & & \\ -1 & 0 & & 0 \\ \hline & & 0 & 1 \\ & 0 & -1 & 0 \end{array} \right) \tag{85}$$

The spinor representation of $SO(5,1)$ on Dirac spinors $\Psi = \begin{pmatrix} \lambda_a \\ \pi^{\dot{b}} \end{pmatrix}$ is generated by

$$\begin{aligned} \Sigma_{AB} = -\tfrac{1}{4}\left[\Gamma_A, \Gamma_B\right] = \begin{pmatrix} (\sigma_{AB})_a{}^b & 0 \\ 0 & (\widetilde{\sigma}_{AB})^{\dot{a}}{}_{\dot{b}} \end{pmatrix} = \\ = \begin{pmatrix} -\frac{1}{4}\left(\sigma_{A a\dot{a}}\widetilde{\sigma}_B{}^{\dot{a}b} - \sigma_{B a\dot{a}}\widetilde{\sigma}_A{}^{\dot{a}b}\right) & 0 \\ 0 & -\frac{1}{4}\left(\widetilde{\sigma}_A{}^{\dot{a}b}\sigma_{B b\dot{b}} - \widetilde{\sigma}_B{}^{\dot{a}b}\sigma_{A b\dot{b}}\right) \end{pmatrix} \end{aligned} \tag{86}$$

The representation is decomposed into two irreducible ones corresponding to the left- and right-handed Weyl spinors. It turns out that the representation (86) and its complex conjugated are equivalent: $(\sigma^*_{AB})_{\dot{a}}{}^{\dot{b}} = I_{\dot{a}}{}^a(\sigma_{AB})_a{}^b I_b{}^{\dot{b}}$, $(\widetilde{\sigma}^*_{AB})^a{}_b = \widetilde{I}^a{}_{\dot{a}}(\widetilde{\sigma}_{AB})^{\dot{a}}{}_{\dot{b}}\widetilde{I}^{\dot{b}}{}_b$. So, one can convert the dotted spinor indices into undotted ones

$$\overline{\lambda}_a = I_a{}^{\dot{a}} \overset{*}{\lambda}_{\dot{a}} \quad , \qquad \overline{\pi}^a = \widetilde{I}^a{}_{\dot{a}} \overset{*}{\pi}{}^{\dot{a}}$$

While the gradient and contragradient representations are inequivalent because of absence of an object raising and/or lowering spinor indices as distinguished from the four-dimensional case. It is convenient to turn from the matrices $(\sigma_A)_{a\dot{a}}, (\widetilde{\sigma}_A)^{\dot{a}a}$ to $(\sigma_A)_{ab} = (\sigma_A)_{a\dot{a}}\widetilde{I}^{\dot{a}}{}_b, (\widetilde{\sigma}_A)^{ab} = \widetilde{I}^a{}_{\dot{a}}(\widetilde{\sigma}_A)^{\dot{a}a}$. They possess a number of relations

$$\begin{gathered} (\sigma_A)_{ab} = -(\sigma_A)_{ba} \quad (\widetilde{\sigma}_A)^{ab} = -(\widetilde{\sigma}_A)^{ba} \\ (\sigma_A)_{ab}(\sigma^A)_{cd} = -2\epsilon_{abcd} \quad (\widetilde{\sigma}_A)^{ab}(\widetilde{\sigma}^A)^{cd} = -2\epsilon^{abcd} \\ (\sigma_A)_{ab} = -\tfrac{1}{2}\epsilon_{abcd}(\widetilde{\sigma}_A)^{cd} \quad (\widetilde{\sigma}_A)^{ab} = -\tfrac{1}{2}\epsilon^{abcd}(\sigma^A)_{cd} \\ (\sigma_A)_{ab}(\widetilde{\sigma}^A)^{cd} = 2\left(\delta_a{}^c\delta_b{}^d - \delta_a{}^d\delta_b{}^c\right) \quad , \quad (\sigma_A)_{ab}(\widetilde{\sigma}_B)^{ba} = -4\eta_{AB} \\ (\sigma_A)_{ab}(\widetilde{\sigma}_B)^{bc} + (\sigma_B)_{ab}(\widetilde{\sigma}_A)^{bc} = -2\eta_{AB}\delta_a{}^c \\ (\widetilde{\sigma}_A)^{ab}(\sigma_B)_{bc} + (\widetilde{\sigma}_B)^{ab}(\sigma_A)_{bc} = -2\eta_{AB}\delta^a{}_c \end{gathered} \tag{87}$$

Here we introduced two invariant tensors ϵ_{abcd} and ϵ^{abcd}, totally antisymmetric in indices and $\epsilon_{1234} = \epsilon^{1234} = 1$. With the aid of introduced objects one may convert the vector indices into antisymmetric pairs of spinor ones. E.g. for a given vector p_A

$$p_A \to p_{ab} = p_A(\sigma^A)_{ab} \ , \quad p^{ab} = p_A(\widetilde{\sigma}^A)^{ab} \ , \quad p_A = -\frac{1}{4}p_{ab}(\widetilde{\sigma}_A)^{ba} = -\frac{1}{4}p^{ab}(\sigma_A)_{ba} \tag{88}$$

Consider two objects

$$(\sigma_{ABC})_{ab} = \frac{1}{4}(\sigma_A\widetilde{\sigma}_B\sigma_C - \sigma_A\widetilde{\sigma}_B\sigma_C)_{ab} \ , \quad (\widetilde{\sigma}_{ABC})^{ab} = \frac{1}{4}(\widetilde{\sigma}_A\sigma_B\widetilde{\sigma}_C - \widetilde{\sigma}_A\sigma_B\widetilde{\sigma}_C)^{ab} \tag{89}$$

They obey the following properties:

$$\begin{gathered}
(\sigma_{ABC})_{ab} = (\sigma_{ABC})_{ba} \quad (\widetilde{\sigma}_{ABC})^{ab} = (\widetilde{\sigma}_{ABC}).^{ba} \\
(\sigma_{ABC})_{ab} = \tfrac{1}{6}\epsilon_{ABCDEF}\left(\sigma^{DEF}\right)_{ab} \quad (\widetilde{\sigma}_{ABC})^{ab} = -\tfrac{1}{6}\epsilon_{ABCDEF}(\widetilde{\sigma}^{DEF})^{ab} \\
(\sigma_{ABC})_{ab}(\widetilde{\sigma}^{ABC})^{cd} = 6\left(\delta_a{}^c\delta_b{}^d + \delta_a{}^d\delta_b{}^c\right) \\
(\sigma_{ABC})_{ab}\left(\sigma^{ABC}\right)_{cd} = (\widetilde{\sigma}_{ABC})^{ab}(\widetilde{\sigma}^{ABC})^{cd} = 0 \\
(\sigma_{ABC})_{ab}(\widetilde{\sigma}^{DEF})^{ba} = \epsilon_{ABC}{}^{DEF} + \delta_A^{[D}\delta_B^E\delta_C^{F]} \\
(\widetilde{\sigma}_{ABC})^{ab}\left(\sigma^{DEF}\right)_{ba} = -\epsilon_{ABC}{}^{DEF} + \delta_A^{[D}\delta_B^E\delta_C^{F]}
\end{gathered} \tag{90}$$

The brackets around the indices mean antisymmetrization. With the aid of introduced objects any antisymmetric Lorentz tensor of the third rank may be converted into a pair of symmetric bispinors.

$$\begin{gathered}
M_{ABC} = \tfrac{1}{12}(M^{ab}(\sigma_{ABC})_{ba} + M_{ab}(\widetilde{\sigma}_{ABC})^{ba}) \\
M^{ab} = M^{ABC}(\widetilde{\sigma}_{ABC})^{ab} \ , \quad M_{ab} = M^{ABC}(\sigma_{ABC})_{ab}
\end{gathered} \tag{91}$$

In conclusion we write out the Fierz identities and rules of complex conjugation for different spinor bilinears. For the sake of simplicity we omit the contracted spinor indices throughout this paper, e. g. $(\chi\widetilde{\sigma}_A\psi) = \chi_a(\widetilde{\sigma}_A)^{ab}\psi_b, (\chi\widetilde{\sigma}_{ABC}\psi) = \chi_a(\widetilde{\sigma}_{ABC})^{ab}\psi_b$

$$\begin{gathered}
\psi_a\chi_b = \tfrac{1}{4}(\psi\widetilde{\sigma}_A\chi)\sigma^A{}_{ab} + \tfrac{1}{12}(\psi\widetilde{\sigma}_{ABC}\chi)\left(\sigma^{ABC}\right)_{ab} \\
\chi^b\psi_a = \tfrac{1}{4}(\chi\psi)\,\delta_a{}^b - \tfrac{1}{2}(\chi\sigma_{AB}\psi)\left(\sigma^{AB}\right)_a{}^b \\
(\psi\chi)^* = \left(\overline{\psi}\overline{\chi}\right) \quad , \qquad (\psi\overline{\chi})^* = -\left(\overline{\psi}\chi\right) \\
(\chi\widetilde{\sigma}_A\psi)^* = (\overline{\chi}\widetilde{\sigma}_A\overline{\psi}) \quad , \qquad (\overline{\chi}\widetilde{\sigma}_A\psi)^* = -(\chi\widetilde{\sigma}_A\overline{\psi}) \\
(\overline{\chi}\widetilde{\sigma}_{ABC}\psi)^* = -(\chi\widetilde{\sigma}_{ABC}\overline{\psi}) \ , \quad (\chi\widetilde{\sigma}_{ABC}\psi)^* = (\overline{\chi}\widetilde{\sigma}_{ABC}\overline{\psi})
\end{gathered} \tag{92}$$

Analogous relations take place for spinor with upper indices.

References

[1] Frenkel J., Z. für Physik **37** (1926), 243; I. E. Tamm, V. L. Ginzburg, JETP **17** (1947), 227; Barut A. O., "*Electrodynamics and Classical Theory of Fields and Particles*", MacMillan New York 1964; Frydryszak A., "*Lagrangian models of particles with spin: the first seventy years*", hep-th/9601020.

[2] Berezin F. A., Marinov M. S., Ann. Phys. **104** (1977) 336; A. Barducci, R. Casalbuoni and L. Lusanna, Nuovo Cim. **A35**, 377 (1976); L. Brink, S. Deser, B. Zumino, P. Di Veccia and P. Howe, Phys. Lett. **B64** (1976) 435; V. D. Gershun and V. I. Tkach, JETP Lett., **29** (1979) 288; P. Howe, S. Penati, M. Pernicci and P. Townsend, Phys. Lett. **215** (1988) 555; Class. Quantum Grav. **6** (1988) 1125.

[3] A. O. Barut and N. Zanghi, Phys. Rev. Lett. **52**, 2009 (1984); A. P. Balachandran, G. Marmo, A. Stern, and B.-S. Skagerstam, *Gauge Symmetries and Fibre Bundles: Application to Particle Dynamics*, Lecture Notes in Physics, vol.188 (Springer, 1983); R. Marnellius U. Mårtensson, Nucl. Phys. **B335**, 395 (1991); Int. J. Mod. Phys. **A6**, 807 (1991).

[4] S. M. Kuzenko, S. L. Lyakhovich and A. Yu. Segal, Int. J. Mod. Phys. **A10**, 1529 (1995); Phys. Lett. **B348**, 421 (1995); S. M. Kuzenko, S.L.Lyakhovich, A. Yu. Segal and A. A. Sharapov, "*Massive spinning particle on anti-de Sitter space*", hep-th/9509062, to appear in Int. J. Mod. Phys.**A** (1996).

[5] S.L.Lyakhovich, A. Yu. Segal and A. A. Sharapov, "*A universal model of D=4 spinning particle*", hep-th/9603174, to appear in Phys Rev **D** (1996).

[6] A. A. Kirillov, *Elements of the theory of group representations* (Springer-Verlag, Berlin, 1976); B. Kostant, Lect. Notes in Math. **170** (1970), 87; J. M. Souriau, *Structure des Systemes Dynamiques* (Dunod, Paris, 1970).

[7] M. B. Green and J. H. Schwarz, Phys. Lett. **136B** (1984), 367; M. B. Green, J. H. Schwarz and E. Witten, *Superstring Theory* (Cambridge University Press, 1987).

[8] T. Kugo and P. Townsend, Nucl. Phys. **B221** (1983), 357; A. Sudberry J. Phys. **A17** (1984), 939.

[9] M. Cederwall *Introduction to the division algebras, sphere algebras and twistors*, Goeteborg Preprint, hep-th/9310115.